Last time: we used Sylow theorems 1,2,3 to classify all groups of order

- p : $\mathbb{Z}/p\mathbb{Z}$
- pq s.t. $p \nmid q-1$, $q \nmid p-1$: $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$
- p^2 : $\mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$
- p^2q s.t. $p \nmid q-1$, $q \nmid p^2-1$: $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$
- 12 : $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ or

  A_4 or D_{12} or dicyclic group $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$
- simple of order 60 : must be isomorphic to A_5

Today: recall that any finite abelian group was a direct product of $\underbrace{\text{abelian } p\text{-groups}}_{\cong \mathbb{Z}/p^{k_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p^{k_n}\mathbb{Z}}$;

we will show that any finite nilpotent group is a direct product of p-groups (its Sylow subgroups)

THM: let G be a finite group; G is nilpotent $\iff$

$$\iff \quad G \cong P_1 \times \dots\dots \times P_k$$

where P_i is a p_i-group, and $p_1, \dots, p_k$ are distinct primes

order $p_1^{d_1} \cdots p_k^{d_k}$ order $p_1^{d_1}$ order $p_k^{d_k}$

Note: each P_i is a normal Sylow p_i-subgroup of G

$$P_i = e \times \dots \times e \times P_i \times e \times \dots \times e \ni h = (e, \dots, e, h, e, \dots, e)$$

$$P_1 \times \dots\dots\dots \times P_k \ni g = (g_1, \dots, g_k)$$

$$ghg^{-1} = (g_1 e g_1^{-1}, \dots, g_{i-1} e g_{i-1}^{-1}, g_i h g_i^{-1}, g_{i+1} e g_{i+1}^{-1}, \dots, g_k e g_k^{-1}) = (e, \dots, e, h', e, \dots, e)$$

$$\Downarrow$$

$$P_i = e \times \dots \times e \times P_i \times e \times \dots \times e \quad \text{is normal in } G$$

Theorem is equivalent to the following fact: a finite group G is nilpotent $\iff$ it has a unique (normal) Sylow p-subgroup $\forall p$

Recall that G is nilpotent if its lower central series

$$\cdots \trianglelefteq G^{\{2\}} \trianglelefteq G^{\{1\}} \trianglelefteq G^{\{0\}} = G$$

where $G^{\{i\}} = [G^{\{i-1\}}, G]$ eventually has $G^{\{k\}} = 1$ for some k

(if this happens, then these inclusions must be strict because if $G^{\{i-1\}} = G^{\{i\}}$, then $G^{\{i+1\}} = [G^{\{i\}}, G] = [G^{\{i-1\}}, G] = G^{\{i\}}$)

<u>Prop</u>: a group G is nilpotent $\iff$ it has a central series

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{k-1} \triangleleft G_k = G$$

s.t. $G_i/G_{i-1} \leq Z(G/G_{i-1})$ and every G_i is normal in G

$\Updownarrow$

<u>Proof</u>: $[g][h][g]^{-1}[h]^{-1} = e \mod G_{i-1}$, $\forall g \in G, \forall h \in G_i$

$\Updownarrow$

$[ghg^{-1}h^{-1}] = e \mod G_{i-1}$

$\Updownarrow$

$ghg^{-1}h^{-1} \in G_{i-1} \iff [G_i, G] \leq G_{i-1}$

"$\Longrightarrow$" G is nilpotent says that $G^{\{k\}} = 1$ for some k

$G_i = G^{\{k-i\}}$ will give you a central series

"$\Longleftarrow$" assume $\exists$ a central series, i.e. $[G_i, G] \leq G_{i-1}$ $\forall i$

claim $G^{\{k-i\}} \leq G_i$, $\forall i$

prove by descending induction on i

$i=K$: $\quad G^{\{0\}} = G = G_K$

assume $\quad G^{\{K-i\}} \le G_i \quad$ and prove that $\quad G^{\{K-i+1\}} \le G_{i-1}$

$$G^{\{K-i+1\}} := [G^{\{K-i\}}, G] \le [G_i, G] \le G_{i-1} \qquad \square$$

Prop: any subgroup of a nilpotent group is nilpotent

Prop: any quotient of a nilpotent group is nilpotent

Prop: any central extension of nilpotent groups is nilpotent

$$1 \to K \to G \to L \to 1 \quad \text{s.t} \quad K \le Z(G)$$

L nilpotent implies G nilpotent (K is abelian, hence nilpotent)

Prop: G, G' are nilpotent $\Rightarrow$ $G \times G'$ is nilpotent

$\Uparrow$

$$(G \times G')^{\{i\}} = G^{\{i\}} \times G'^{\{i\}}$$

prove by induction on i
- base case $i=0$, both groups are $G \times G'$
- induction step $(G \times G')^{\{i-1\}} = G^{\{i-1\}} \times G'^{\{i-1\}}$

$$(G\times G')^{\{i\}} = \left[(G\times G')^{\{i-1\}}, G\times G'\right]$$

= generated by commutators $(\underset{G^{\{i-1\}}}{g},\underset{G'^{\{i-1\}}}{g'})(\underset{G}{h},\underset{G}{h'})(g,g')^{-1}(h,h')^{-1}$

= generated by commutators $(\underbrace{ghg^{-1}h^{-1}}_{G^{\{i\}}=[G^{\{i-1\}},G]},\ \underbrace{g'h'g'^{-1}h'^{-1}}_{G'^{\{i\}}=[G'^{\{i-1\}},G']})$

$= G^{\{i\}}\times G'^{\{i\}}$

Thm: any finite p-group P is nilpotent

Proof: if $P\neq 1$, then $Z(P)\neq 1$

$$1\to Z(P)\to P\to P/Z(P)\to 1$$

by induction on $|P|$, this guy is nilpotent
also $Z(P)$ is abelian, hence nilpotent
$\Big\}\overset{\text{Prop}}{\Longrightarrow}$ P is nilpotent $\square$

any p-group is nilpotent
any direct product of nilp is nilp
$\Big\}\Longrightarrow$ $P_1\times\ldots\times P_k$ are nilpotent
$\Downarrow$
"if" statement of main THM

Def: a group G has the normalizer property if the normalizers of proper subgroups strictly grow, i.e.

$$\forall\ H < G \text{ , we have } H < N_G(H)$$

G is nilpotent $\Longrightarrow$ G has the normalizer property

if G is finite

if G is finite

$$G \cong P_1 \times \dots \times P_k$$

Proof of $\Longrightarrow$: statement about a slide above

Proof of $\Longrightarrow$: let H be any proper subgroup of G

take $i \geq 0$ s.t. $G^{\{i\}} \leq H$ (such i exists because G nilpotent)

assume H fails normalizer property, i.e. $H = N_G(H)$

Claim: $G^{\{i-1\}} \leq N_G(H)$; why?

$\forall\ \substack{g \in G^{\{i-1\}} \\ h \in H}$, we have $ghg^{-1}h^{-1} \in [G^{\{i-1\}}, G] = G^{\{i\}} \leq H$

$\Downarrow$

$ghg^{-1} \in H \Longrightarrow G^{\{i-1\}} \leq N_G(H)$

$G^{\{i-1\}} \leq H$

$\Downarrow$

$G^{\{i-2\}} \leq H$

$\Downarrow$

$\vdots$

$\Downarrow$

$G = G^{\{0\}} \leq H$ impossible because H is a proper subgroup.

Proof of $\Longrightarrow$: G is a finite group with normalizer property; let $P_1, \dots, P_k$ be Sylow subgroups for primes dividing G

Lemma: if P is a Sylow p-subgroup of G,
(TBP later) then $N_G(N_G(P)) = N_G(P)$

Because G satisfies the normalizer property, Lemma implies

$$N_G(P_1) = \dots = N_G(P_k) = G$$

$$\Downarrow$$

$P_1, \dots, P_k$ are normal Sylow subgroups

$$P_1 \times \dots \times P_k \xrightarrow{f} G \quad , \quad f((g_1, \dots, g_k)) = g_1 \cdots g_k$$

• f is a homomorphism: $g_1 \cdots g_k \, g_1' \cdots g_k' = g_1 g_1' \cdots g_k g_k'$
follows from the fact that $[P_i, P_j] = \{e\}$ in G $\forall i \neq j$

$[P_i, P_j] \leq P_i$ because P_i normal $\Longrightarrow$ is a subgroup of P_i and P_j, which have coprime orders, hence $[P_i, P_j]$ has order 1

$ghg^{-1}h^{-1}$, $g \in P_i, h \in P_j \Rightarrow hgh^{-1} \in P_i$

• f is injective: assume $f((g_1, \ldots, g_k)) = g_1 \cdots g_k = e$
P_1 P_k

$\Downarrow$

$g_k = (g_1 \cdots g_{k-1})^{-1} \Rightarrow g_k = e$; similarly $g_1 = \cdots = g_{k-1} = e$

order is $\in P_k^{\mathbb{N}}$ order is $\in P_1^{\mathbb{N}} \cdots P_{k-1}^{\mathbb{N}}$

• f is surjective because f is injective and $|P_1 \times \cdots \times P_k| = |G|$.

Lemma: if P is a Sylow p-subgroup of G,
then $N_G(N_G(P)) = N_G(P)$

Proof: $H = N_G(P)$; assume $N_G(H) > H$
pick $g \in N_G(H) \setminus H$ contradiction

gPg^{-1} is also a Sylow p-subgroup of $H \leq G$

because $P \leq H$ and $gHg^{-1} = H$

$\Downarrow$ Sylow 2

$gPg^{-1} = hPh^{-1}$ for some $h \in H$

$\Downarrow$

$$(h^{-1}g)P(h^{-1}g)^{-1} = P \Rightarrow h^{-1}g \in N_G(P) = H \Rightarrow g \in H$$